

What does a strongly excited 't Hooft-Polyakov magnetic monopole do?

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The time evolution of strongly excited $SU(2)$ Bogomolny-Prasad-Sommerfield (BPS) magnetic monopoles in Minkowski spacetime is investigated by means of numerical simulations based on the technique of conformal compactification and on the use of hyperboloidal initial value problem. It is found that an initially static monopole does not radiate the entire energy of the exciting pulse toward future null infinity. Rather, a long-lasting quasi-stable 'breathing state' develops in the central region and certain expanding shell structures – built up by very high frequency oscillations – are formed in the far away region.

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The investigation of solitons in particle physics is of fundamental interest (see e.g. [1] for a recent review). In particular, considerable attention has been paid to the study of 't Hooft-Polyakov magnetic monopole solutions of coupled Yang-Mills-Higgs (YMH) systems [2, 3]. However, the relevant investigations have been almost exclusively restricted to time independent configurations. Thereby it is of obvious physical interest to study dynamical properties of magnetic monopole configurations.

This paper is to report about the results of investigations concerning the time evolution of a strongly excited spherically symmetric $SU(2)$ BPS magnetic monopole [4, 5] on a fixed Minkowski background spacetime by means of numerical techniques. In particular, the underlying Yang-Mills-Higgs system is chosen so that the Yang-Mills field is massive while the Higgs field is massless. The dynamics starts by hitting the static monopole by a concentrated pulse. First the original pulse splits into two, a direct outgoing one and another one going through the origin. Both pulses travel along null geodesics taking away part of the energy of the excitation toward future null infinity, \mathcal{I}^+ , with the help of the massless Higgs field. It is found, however, that this way the excited monopole releases only about half of the energy received. The rest of the energy of the original pulse seems to be restrained by the monopole in accordance of which it develops a long lasting quasi-stable 'breathing state' in the central region and certain expanding shell structures in the far away region. The frequency characterizing the breathing state varies in time and it approaches asymptotically the value of the vector boson mass from below. In the far away region, where the Yang-Mills and the Higgs fields are practically decoupled, the massless Higgs field does the boring job of transporting the energy released gradually by the central monopole to \mathcal{I}^+ , while the behavior of the massive Yang-Mills field in the far away asymptotic region can be characterized by the formation of certain expanding shell structures where all the shells are built up by very high frequency oscillations. These oscillations are found to be modulated by the product of a simple time decaying factor of the form $t^{-1/2}$ and of an essentially self-similar expansion.

The time decay of the examined quantities shows certain type of *universality*. The total energy associated with

the hyperboloidal hypersurfaces decays in time with power $-2/3$, while the amplitude of the oscillating fields decay with power $-5/6$. In a recent work by Forgács and Volkov [6], based on the use of a linear approximation of the BPS monopole, explanation is provided for these universalities.

The investigated dynamical magnetic monopole is described as a coupled $SU(2)$ YMH system. The Yang-Mills field is represented by an $\mathfrak{su}(2)$ -valued vector potential A_a and the associated 2-form field F_{ab} reads as

$$F_{ab} = \nabla_a A_b - \nabla_b A_a + ig[A_a, A_b] \quad (1)$$

where $[,]$ denotes the product in $\mathfrak{su}(2)$ and g stands for the gauge coupling constant. The Higgs field (in the adjoint representation) is given by an $\mathfrak{su}(2)$ -valued function ψ while its gauge covariant derivative reads as $\mathcal{D}_a \psi = \nabla_a \psi + ig[A_a, \psi]$. The dynamics of the investigated YMH system is determined by the action

$$S = \int \left[Tr(F_{ef} F^{ef}) + 2Tr(\mathcal{D}_e \psi \mathcal{D}^e \psi) \right] \epsilon, \quad (2)$$

where ϵ is the 4-dimensional volume element.

Our considerations were restricted to spherically symmetric configurations yielded by the 'minimal' dynamical generalization of the static 't Hooft-Polyakov magnetic monopole configurations [2, 3] (see also [7]). Accordingly, the evolution took place on Minkowski spacetime the line element of which, in spherical coordinates (t, r, θ, ϕ) , is $ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$, while the Yang-Mills and Higgs fields, in the so called *abelian gauge*, were assumed to possess the form

$$A_a = -\frac{1}{g} [w \{ \tau_2 (d\theta)_a - \tau_1 \sin \theta (d\phi)_a \} + \tau_3 \cos \theta (d\phi)_a] \quad (3)$$

$$\psi = H \tau_3, \quad (4)$$

where the generators $\{\tau_I\}$ ($I=1,2,3$) of $\mathfrak{su}(2)$ are related to the Pauli matrices σ_I as $\tau_I = \frac{1}{2}\sigma_I$, moreover, w and H were assumed to be smooth functions of t and r .

The field equations relevant for this system are

$$r^2 \partial_r^2 w - r^2 \partial_t^2 w = w [(w^2 - 1) + g^2 r^2 H^2] \quad (5)$$

$$r^2 \partial_r^2 H + 2r \partial_r H - r^2 \partial_t^2 H = 2w^2 H. \quad (6)$$

The only known analytic solution to (5) and (6) is the static BPS monopole [4, 5]

$$w_0 = \frac{gCr}{\sinh(gCr)}, \quad H_0 = C \left[\frac{1}{\tanh(gCr)} - \frac{1}{gCr} \right], \quad (7)$$

where C is an arbitrary positive constant. Since this solution is stable it was used first to check the efficiency of our numerical code. Later we considered the complete non-linear evolution of a system yielded by strong impulse type excitations of this monopole. All the results below concerns the evolution of such an excited BPS monopole.

The only scale parameter of the above described system is the vector boson mass $m_w = gH_\infty$, where the limit value $H_\infty = \lim_{r \rightarrow \infty} H$ of the Higgs field can in general be shown to be time independent [8]. Since in the case considered here $H_\infty = C \neq 0$, without loss of generality, the parameter choice $g = H_\infty = m_w = 1$ can be ensured to be satisfied by making use of standard rescalings.

To have a computational grid covering the full physical spacetime – ensuring thereby that the outer grid boundary will not have an effect on the time evolution – the technique of conformal compactification, along with the hyperboloidal initial value problem, was used. This way it was possible to study the asymptotic behavior of the fields close to future null infinity, as well as, the inner region for considerably long physical time intervals.

The conformal transformation we used is a slight modification of the static hyperboloidal conformal transformation applied by Moncrief [9]. It is defined by introducing first the new coordinates T and R instead of t and r as

$$T = \omega t - \sqrt{\omega^2 r^2 + 1} \quad \text{and} \quad R = \frac{\sqrt{\omega^2 r^2 + 1} - 1}{\omega r}, \quad (8)$$

where ω is an arbitrary positive constant. The Minkowski spacetime is covered by the coordinate domain given by the inequalities $-\infty < T < +\infty$ and $0 \leq R < 1$. Then the conformally rescaled metric can be given as $\tilde{g}_{ab} = \Omega^2 g_{ab}$, where the conformal factor is $\Omega = \omega(1 - R^2)/2$. The $R = 1$ coordinate line represents \mathcal{I}^+ through which the conformally rescaled metric \tilde{g}_{ab} smoothly extends to the coordinate domain with $R > 1$.

Using the substitution $H(t, r) = h(t, r)/r + H_\infty$ the field equations, (5) and (6), in the new coordinates read as

$$\mathfrak{P}w = w \left[(w^2 - 1) + g^2 (h + H_\infty R \Omega^{-1})^2 \right] \quad (9)$$

$$\mathfrak{P}h = 2 (h + H_\infty R \Omega^{-1}) w^2, \quad (10)$$

where the differential operator \mathfrak{P} is defined as

$$\mathfrak{P} = \frac{4R^2}{(R^2 + 1)^2} \left[\frac{\Omega^2}{\omega^2} \partial_R^2 - \partial_T^2 - 2R \partial_R \partial_T - \frac{2\Omega}{\omega(R^2 + 1)} \partial_T - \frac{\Omega R(R^2 + 3)}{\omega(R^2 + 1)} \partial_R \right]. \quad (11)$$

These equations can be put into the form of a first order strongly hyperbolic system [8]. The initial value problem for such a system is known to be well-posed [10]. In particular, we solved this first order system numerically by making use of the ‘method of line’ in a fourth order Runge-Kutta scheme following the recipes proposed by Gustafsson *et al* [10]. All the details related to the numerical

approach, including representations of derivatives, treatment of the grid boundaries are to be published in [8]. The convergence tests justified that our code provides a fourth order representation of the selected evolution equations. Moreover, the monitoring of the energy conservation and the preservation of the constraint equations, along with the coincidence between the field values which can be deduced by making use of the Green’s function and by the adaptation of our numerical code to the case of massive Klein-Gordon fields, made it apparent that the phenomena described below have to be, in fact, physical properties of the magnetic monopoles.

In each of the numerical simulations initial data on the $T = 0$ hypersurface was specified for the system of our first order evolution equations. In particular, a superposition of the data associated with the BPS monopole, (7), and of an additional pulse of the form

$$(\partial_T w)_o = \begin{cases} c \exp \left[\frac{d}{(r-a)^2 - b^2} \right], & \text{if } r \in [a-b, a+b]; \\ 0, & \text{otherwise,} \end{cases} \quad (12)$$

with $a \geq b > 0$, which is a smooth function of compact support, was used. This choice, providing non-zero time derivative for w , corresponds to ‘‘hitting’’ the static monopole configuration between two concentric shells at $r = a - b$ and $r = a + b$ with a bell shape distribution. Basically the same type of evolution occurs when instead of $(\partial_T w)_o$ we prescribe $(\partial_T h)_o$ in a similar fashion.

For the sake of brevity, all the simulations shown below refer to the same pulse (12) corresponding to the choice of the parameters $a = 2$, $b = 1.5$, $c = 70$, $d = 10$ and $\omega = 0.05$. The energy of this pulse is 55.4% of the static monopole. Hence, the yielded dynamical system cannot be considered as being merely a simple perturbation of the static monopole. We would also like to emphasize that the figures shown below are typical in the sense that for a wide range of the parameters characterizing the exciting pulse, qualitatively, and in certain cases even quantitatively, the same type of responses are produced by the monopole [8].

To start off consider first the product of the energy density ε and $4\pi r^2$ plotted on succeeding $T = \text{const}$ hypersurfaces, providing thereby a spacetime picture of the time evolution (see Fig. 1). The use of $4\pi r^2 \varepsilon$, instead of ε , makes it easier to see the main characteristics of the dynamics up to \mathcal{I}^+ . In the early part the direct energy transport to \mathcal{I}^+ by the Higgs field, with the velocity of light, is apparent. The developing ‘breathing state’ of the monopole and the formation of the expanding shells of high frequency oscillations are both clearly manifested.

The energy radiated to \mathcal{I}^+ can be pictured by plotting (see Fig. 2) the product of the energy current density S and $4\pi r^2$ against time T at \mathcal{I}^+ ($R = 1$). It is apparent that the arrival of the two pulses is followed by a small scale but systematic energy loss of the system which has exactly the same characteristic period as the inner breathing state of the monopole. From the logarithmic plot the asymptotic behavior $4\pi r^2 S_{\text{asympt}} \approx C_S T^{-\gamma_S}$ can be read off where $\gamma_S \approx 1.66$. In virtue of the energy conservation this relation implies that the energy associated with the $T = \text{const}$ hypersurfaces approaches to the energy of the asymptotic final state as $T^{-2/3}$, in agreement with [6].

By inspection of the evolution of the field variables w

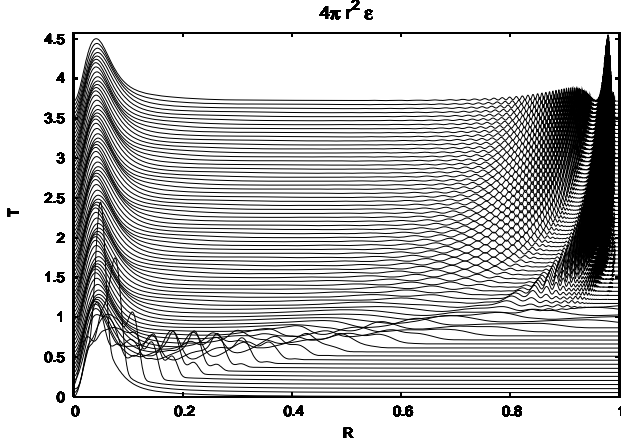


FIG. 1: Spacetime diagram showing the time evolution of the energy density associated with shells of radius r , i.e. of $4\pi r^2 \epsilon$.

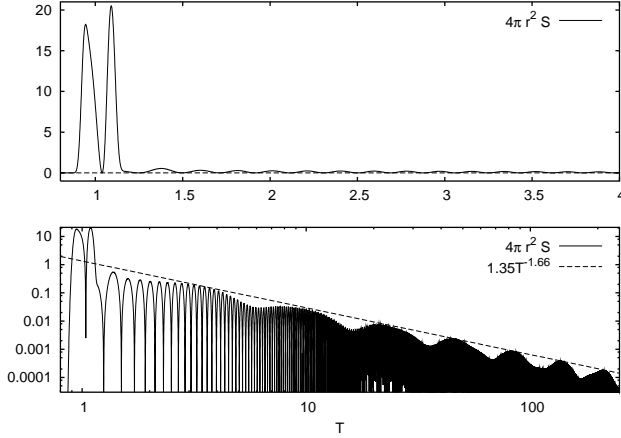


FIG. 2: The time dependence of the product of the energy current density S and $4\pi r^2$ at \mathcal{I}^+ is shown on the intervals $0.8 \leq T \leq 4$ and $0.8 \leq T \leq 250$.

and h it is obvious that the expanding high frequency oscillations are associated with the massive Yang-Mills field exclusively. Fig. 3 shows a constant time slice at $T = 1.695$ of the evolution of $w - w_0$. The formation of the shells built up by high frequency oscillations is transparent on Fig. 3, which is reminiscent of Figs. 1. and 2. of [11]. The behavior of these oscillations in the asymptotic region can be explained by referring to results of [11] where this phenomenon has already been found to characterize the evolution of the simplest linear massive scalar field. In doing this, notice first that by (5) w/r satisfies asymptotically the massive Klein-Gordon equation. This, along with the claims of [11], implies then that the oscillations are modulated by two factors. By an overall factor $t^{-1/2}$ scaling down the oscillations in time, moreover, by an essentially self-similar expansion, i.e. by a function depending on t and r only in the combination $\rho = r/t$.

Probably, the most interesting unexpected feature of the time evolution is the appearance of the breathing state of the monopole (see the central region of Fig. 1). To have a quantitative characterization of this phenomenon it is

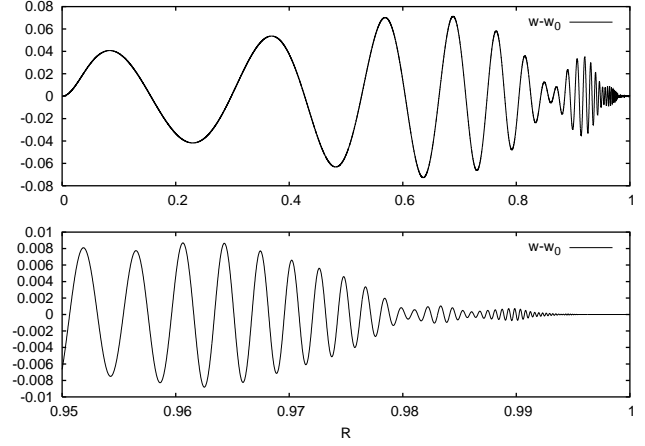


FIG. 3: The difference $w - w_0$ is plotted on time slice $T = 1.695$. The formation of the high frequency oscillations sorted in certain shell structures is transparent.

informative to consider a constant R slice of the deviation $\epsilon - \epsilon_0$ of the full energy density ϵ and that of the static monopole ϵ_0 . The breathing state of the monopole can

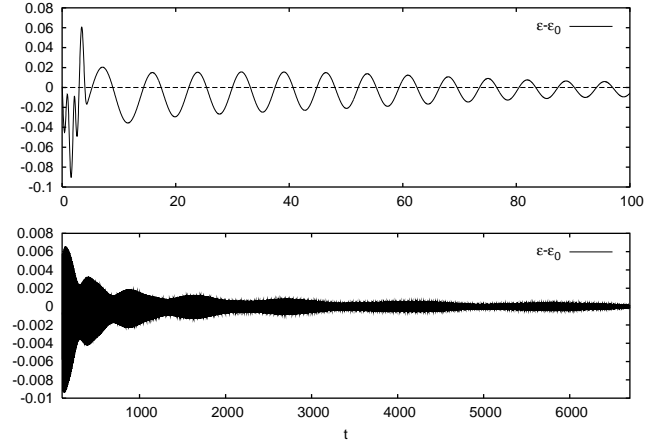


FIG. 4: The deviation $\epsilon - \epsilon_0$ of the full energy density ϵ from that of the static monopole ϵ_0 is plotted against the physical time t for intervals $0 \leq t \leq 100$ and $100 \leq t \leq 6700$ at $R = 0.0254$. It is striking how quickly the breathing state develops.

be characterized by the time dependence of the amplitude and the frequency of the associated quasi-normal oscillations, as well as, by the power spectrum of the oscillations. The time dependence of the amplitude is shown by Fig. 4. Note that the center of the oscillations is actually lower than ϵ_0 , which implies that the time average of the energy contained in the central region is smaller than the energy contained in the same region of the static monopole. This is most likely due to nonlinear effects.

The time dependence of the frequency and the amplitude of the oscillations was determined by fitting a simple function of the form $\epsilon - \epsilon_0 = a \sin(\omega t + b) + c$ to the numerical data on a sufficiently short time interval so that this interval was shifted point by point through the entire time evolution. The resulted graph is shown

by Fig. 5. The frequency of the oscillations is essentially

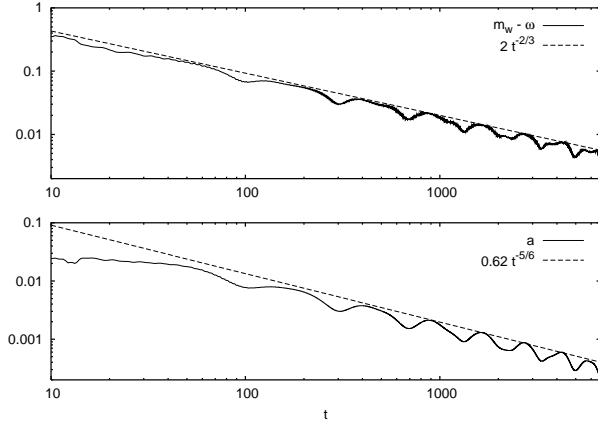


FIG. 5: The frequency difference $m_w - \omega$ and the amplitude a of the oscillations of the breathing state, shown by Fig. 4, is plotted against the physical time t .

increasing, asymptotically approaching the value of the vector boson mass m_w . The logarithmic plot suggests a relatively simple form for the overall behavior of the frequency-time function. In particular, the approximate relation $\omega \approx m_w - C_\omega t^{-\gamma_\omega}$ seems to be valid not merely asymptotically but for the entire evolution, where γ_ω was found up to a high accuracy to be $2/3$. Exactly the same value $\gamma_\omega = 2/3$ was found in [6]. The asymptotic behavior of the amplitude of the oscillations can be approximated by the simple form $a_{asympt} \approx C_a t^{-\gamma_a}$, where $\gamma_a \approx 0.833$, which is in good agreement with the value of $-5/6$ of [6]. The energy contained in some finite radius is described by the time average of the energy, i.e. by the function c , not by the amplitude a . For this reason there is no contradiction between the exponents $-5/6$ in the central region and $-2/3$ at infinity. The function c appears to be smaller and decays faster than the amplitude, and can also take positive and negative values depending on the location.

It is also of interest to consider the power spectrum $P(\omega; t_1, t_2)$ of the oscillations defined as twice of the absolute value of the Fourier transform $(\varepsilon - \varepsilon_0)(\omega; t_1, t_2) = (2\pi)^{-1/2} \int_{t_1}^{t_2} (\varepsilon - \varepsilon_0)(t) e^{-i\omega t} dt$. Fig. 6 shows $P(\omega; t_1, t_2)$ with t_2 having the fixed value $t_2 = 11107$ and t_1 being chosen to take the values 6.8, 42, 60, 95 and 165, respectively. By varying t_1 it is possible to monitor the change of the frequency of the oscillations in the relevant early period.

These graphs support the perturbative result of Forgács and Volkov [6] claiming that there has to be an infinite family of ‘resonant states’, associated with the breathing state of the monopole, possessing discrete frequencies. See, for instance, the peaks of the graph of $P(\omega; t_1 = 6.8)$ at $\omega \approx 0.844, 0.933, 0.9645, 0.9775, 0.985, \dots$. According to Fig. 6 the lower frequency members of the resonant states die out faster than the higher frequency ones which is consistent with the above mentioned increase of the overall frequency of the breathing state.

In summary, the time evolution of strongly excited BPS monopoles has been studied. It is found to be generic that in the central region a long lasting quasi-periodic breathing state develops. The behavior of all the examined quan-

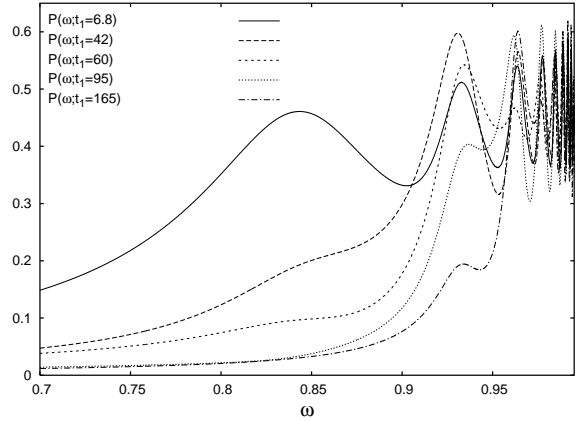


FIG. 6: The power spectrum of the oscillations of $\varepsilon - \varepsilon_0$ at $R = 0.0254$ relevant for the intervals $[t_1, t_2]$, with $t_2 = 11107$ while t_1 taking the values 6.8, 42, 60, 95 and 165 is shown.

ties justifies the intuitive expectation that in the inner region the system settles down to the original static BPS monopole, while the self-similarly expanding oscillations disperse asymptotically in the far away region.

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